

SIGNAL RESTORATION AFTER TRANSMISSION  
THROUGH AN ADVECTIVE AND DIFFUSIVE  
MEDIUM

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# SIGNAL RESTORATION AFTER TRANSMISSION THROUGH AN ADVECTIVE AND DIFFUSIVE MEDIUM

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**ABSTRACT.** Inverse problem, regularisation, singular perturbation, wave splitting, wave propagators, square root operator, inverse mass transport This paper considers an inverse problem associated with mass transport in a pipe. It illustrates how wave splitting techniques can be utilised for an inverse problem associated with one-dimensional mass transport processes. This is done by using a generalisation of Fick's law which introduces a relaxation parameter into the problem, so converting the parabolic partial differential equation by a singular perturbation into a hyperbolic one. This generalised law by ensuring finite mass flux propagation speeds, enables a stable equation to be utilised to reconstruct the interior boundary condition; so providing a regularised solution to the inverse problem. Theoretical results for the solution of the inverse problem are also developed.

## 1. INTRODUCTION

The perfusion apparatus is an experimental *in vitro* tool used by endocrinologists to model information transfer in endocrine systems (McIntosh & McIntosh, 1983; McIntosh, McIntosh, & Kean, 1984; Evans et al., 1985). The major drawback of the perfusion system derives from the dispersion, diffusion and mixing of the hormone within the apparatus which distort the original released hormone concentration profile. Recently mathematical techniques have been applied to improving this situation (Shorten & Wall, 2001).

This paper presents another approach to the inverse problem of concentration signal restoration after signal transmission through an advective and diffusive medium, with applications to perfusion.

The problem considered here has direct application to a related problem involving the estimation of secretion of *adrenocorticotrophic* hormone (ACTH) from the pituitary gland. In this problem, assays of the blood flow are taken downstream from the pituitary in a horse, which is secreting ACTH, and it is then required to estimate the concentration of ACTH at the pituitary, (Alexander, Irvine, Liversey, & Donald, 1988). This *in vivo* experimental technique poses similar problems to the perfusion apparatus mentioned previously, but the flow situation is more complicated.

When a mass concentration of a material is transported within a fluid of a different material, the estimation of the final temporal profile, downstream of the injection point, from the knowledge of the initial injection temporal profile, is a relatively straight-forward problem; it is required

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to solve a well-posed parabolic partial differential equation. This is termed a *direct* problem. However, when a mass concentration of a material flowing and diffusing within an advecting fluid is measured downstream of the injection point, the estimation of the initial injection profile is a difficult problem. This is termed an *inverse* problem; in particular an inverse source problem. This problem is difficult because it is *ill-posed* and besides the difficulty of problem formulation, it must also be made well-posed.

It is assumed throughout this paper that the advective medium also allows diffusion. If this is not the case the problem is considerably simpler, and can be solved by the techniques illustrated in (Wall & Lundstedt, 1998; Connolly & Wall, 1997). In many other applications it is required to estimate an input signal given a signal that has been modified by transmission through a distorting system (Connolly & Wall, 1997; Lundstedt & He, 1994, 1997; Weber, 1981; Eldén, 1988; Murio & Roth, 1988; Wall, 1997).

We now discuss a source reconstruction problem, for the advection equation, in a well known transport situation. If a pollutant, with concentration  $c(x, t)$ , is emitted upstream, at a station  $x = 0$ , in a medium flowing with speed  $\bar{v}$ , and the pollutant concentration is measured downstream, the methods given in this paper can be utilised to estimate the magnitude and temporal distribution of the emission, expressed by  $c(0, t)$ . The model is simplistic, and the problem is considered one-dimensional; in this case the model equation, one is first lead to would be

$$\partial_t c + \bar{v} \partial_x c - \kappa^2 \partial_x^2 c = 0,$$

where  $\kappa^2$  is the fluid diffusivity. As explained above, as this partial differential equation is parabolic, the problem of estimation of  $c(0, t)$ , from a down-stream measurement of  $c(\ell, t)$ , where  $\ell > 0$ , is ill-posed. In this paper, this problem is converted to a well-posed one by studying a related hyperbolic equation.

Weber (Weber, 1981) has converted a parabolic equation into a hyperbolic one in order to produce a well-posed inverse problem, and this idea has been further developed by a number of authors (Eldén, 1988; Murio & Roth, 1988; Wall, 1997; Murio, 1993). Similarly, in this paper, which considers the solution of inverse problems associated with parabolic direct problems, the wave speed of an associated hyperbolic problem is to be considered as a regularisation parameter.

Wave splitting and invariant imbedding techniques have been very successful in their application to many inverse problems for hyperbolic equations. They have also been successful when applied to elliptic problems (Powell, 1995), but they have not been effective for parabolic equations. Indeed it has been shown by (Vogel, 1992) that layer stripping techniques are not suitable for parabolic equations.

The paper (Wall & Olsson, 1997) which is henceforth denoted as (I), is the outcome of research towards the application of wave splitting and invariant imbedding techniques to phenomena that are generally considered parabolic in nature. We concentrate, in the sequel, our ideas towards the evaluation of mass transport processes in fluids, through wave splitting techniques when a generalised form of Fick's law is utilised. This generalised law provides physical motivation for the extension of the direct parabolic problem to a hyperbolic one. The wave splitting method enables us to extend the integral equation method of (Murio & Roth, 1988), as has been done for purely

diffusive processes in (Wall, 1997), to advective problems in which the material parameters are functions of the spatial variable.

In viscous flow problems, involving a solute concentration in a pipe, the variation of the velocity over the cross-section of the pipe is an important factor in the dispersion of the solute; it is called shear dispersion. The dispersion of a pulse of concentration flowing down the pipe is in general due to the combined action of shear dispersion, parallel to the axis of the pipe, and molecular diffusion, predominantly in the radial direction. When the molecular diffusion coefficient within the flow is very small, then shear dispersion is the dominant dispersive effect; this regime is examined in (Shorten & Wall, 1998) for an inverse problem similar to the one considered in this paper. Taylor (Taylor, 1953) has developed a theory which in a certain parameter regime enables the combined effects of shear dispersion and radial molecular diffusion to be replaced by an equivalent one-dimensional advection-diffusion equation. A considerable literature has built up about this direct problem (for recent application of modern theory to this direct problem see (Watt & Roberts, 1995)). The Taylor theory therefore enables an apparent diffusion coefficient to be used to model both the shear and cross-diffusion dispersion effects as a one-dimensional advection-diffusion problem. The theory presented here is therefore again applicable with the molecular diffusion parameter replaced by an appropriate effective value.

When  $\bar{v}$  tends to zero, the reconstruction problem reverts to the one for a pure diffusion problem, as considered in (I). For non-zero  $\bar{v}$  and small  $\kappa$ , the problem may be solved without regularisation by techniques similar to those (Wall & Lundstedt, 1998)<sup>1</sup>. However, for moderate values of  $\kappa$ , techniques which restore the continuity of the solution on the data like those discussed in this paper, must be utilised.

Wave splitting techniques, as used in other problems, proceed by finding an analytic representation of the square root of an operator. This operator is often a differential operator, and the square root is a pseudo-differential operator. In this paper, because of the complex phenomena under consideration, a new feature is that the pseudo-differential operator is related to the factorisation of a quadratic operator equation (see (Karlsson & Rikte, 1998) for another interesting operator factorisation).

In §2 the prerequisite equations are developed for *one-dimensional* mass transport, through hyperbolisation. The homogeneous parameter model and its reduction to various problems is discussed in §3. It is proven in this section, that the hyperbolic case is well conditioned and provides a regularised solution to the parabolic problem. The parabolic problem can be considered as the limiting case of the hyperbolic problem by a singular perturbation. Then in §4, the wave splitting concept is used to transform these equations into two coupled one-way wave equations, with only the second-order differential form covered. In §5, the equations for the wave propagators are derived, and these are specialised to the forward and transmission Green propagators in §5.1 and §5.2, respectively. How these propagators are used in the signal reconstruction problem is discussed in §6.

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<sup>1</sup>Similar techniques can be applied to problems with non-zero  $\kappa$  and large  $\bar{v}$ .

As many multivariate derivative operators are involved in this paper, it is convenient to use a number of commonly used derivative notations for clarity of exposition; these are represented by:

$$\partial_t f(x, t, s) \equiv D_t f(x, t, s) \equiv D_2 f(x, t, s).$$

## 2. PRELIMINARIES

It has been shown that when heat waves are important, the equation connecting the heat flux to the temperature must at least have an extra thermal inertia term added, when compared to Fourier's conduction law. The Cattaneo equation (Cattaneo, 1948) for heat flow in a heat conducting solid has such a term<sup>2</sup>. This leads to the idea of generalising the Fickian law connecting mass flux  $J$ , directed in the  $x$ -coordinate direction, to the mass concentration per unit volume, and this can be written as

$$(2.1) \quad \tau \frac{\partial J}{\partial t} + J = -\kappa^2 \frac{\partial c}{\partial x},$$

where  $\tau$  is a relaxation time, and  $\kappa^2$  is the mass diffusivity of the media. For convenience it is expedient for us to define diffusivity,  $\kappa^2$ , as the square of the usual terminology. The relaxation time depends on the mechanism of mass transport, and represents the time lag needed to establish steady-state mass transfer in an element of volume when a concentration gradient is suddenly applied to that element.

When Fick's law is utilised as the constitutive equation connecting mass flux and the gradient of mass concentration, the resultant equation governing the dynamics of the mass flow is a parabolic equation, and consequently has the non-physical property that information propagates at an infinite speed; this results in the zero propagation time paradox. When such an equation is used in applications such as those modelling spatio-temporal population density distributions it can lead to erroneous densities. It also means that the partial differential equation describing the phenomenon is unilateral with respect to time flow. However parabolic equations do propagate some properties at a finite velocity (Day, 1997a, 1997b; Herrera & Falc3n, 1995). The theory considered in this paper is linear, and can therefore only be considered as appropriate for a small perturbation theory, or alternatively as a linearization of a more general nonlinear theory (Barletta & Zanchini, 1997).

The other equation necessary to link concentration density to the transport mass flux, the velocity field of the embedding medium  $\bar{v}$ ,<sup>3</sup> and the internal rate of production of mass concentration,  $r$ , is the conservation of mass equation

$$\frac{\partial c}{\partial t} + \frac{\partial (J + c\bar{v})}{\partial x} = r.$$

<sup>2</sup>See (I) and the references quoted therein.

<sup>3</sup>The velocity field of the embedding medium advects the mass concentration.

These two linear equations can be written as the system

$$(2.2) \quad \begin{bmatrix} 1 & 0 \\ \bar{v} & 1 \end{bmatrix} \partial_x \begin{bmatrix} c \\ J \end{bmatrix} = \begin{bmatrix} 0 & -\kappa^{-2}(\tau \partial_t + 1) \\ -\partial_t - \partial_x(\bar{v}) & 0 \end{bmatrix} \begin{bmatrix} c \\ J \end{bmatrix} + \begin{bmatrix} 0 \\ r \end{bmatrix},$$

where in what follows, unless stated to the contrary, all coefficients in the partial differential equation will be assumed to be independent of the dependent variables, but functions of the spatial variable  $x$ . The coefficients will further be assumed to be time independent; such an assumption holds for many materials (see for example (Wall & Lundstedt, 1998; Åberg, Kristensson, & Wall, 1996), for problems for which this is not the case). Throughout this paper it is assumed that the material parameters  $\kappa^2$ ,  $\bar{v}$ , and  $\tau$  are continuously differentiable in the region of interest. All parameters are assumed to be positive. The parameters which are essential to the discussion in this paper are the diffusivity  $\kappa^2$ , the relaxation time  $\tau$ , and the mass flux wave slowness  $v$ , with  $v^2 = \kappa^{-2}\tau$ .

As the leading matrix of the system (2.2) is non-singular, these equations can be rewritten as the system

$$(2.3) \quad \partial_x \begin{bmatrix} c \\ J \end{bmatrix} = \begin{bmatrix} 0 & -\kappa^{-2}(\tau \partial_t + 1) \\ -\partial_t - \partial_x(\bar{v}) & \bar{v}\kappa^{-2}(\tau \partial_t + 1) \end{bmatrix} \begin{bmatrix} c \\ J \end{bmatrix} + \begin{bmatrix} 0 \\ r \end{bmatrix},$$

and this modified system is what is considered in the sequel. This system can also be written as a second order partial differential equation, but only when  $\tau$  is a constant (*c.f.* (I) where  $\bar{v} \equiv 0$ ).

$$(2.4) \quad v^2 \partial_t^2 c + \kappa^{-2} \partial_t c - \partial_x^2 c + \kappa^{-2} \bar{v} \partial_x c - 2 \partial_x(\ln(\kappa)) \partial_x c \\ + \kappa^{-2} \partial_x(\bar{v}) c + v^2 \partial_x(\bar{v}) \partial_t c + v^2 \bar{v} \partial_t \partial_x c = \kappa^{-2} (\tau \partial_t r + r).$$

It is considered in the remainder of the paper that  $\tau$  will *not* be  $x$ -dependent if the second-order equation (2.4) is under consideration; this is because this equation is only valid in this case. For the system (2.3), it is *not* necessary to make this assumption. However when the hyperbolic equations are considered as a singular perturbation of the parabolic equations,  $\tau$  is generally thought of as constant.

### 3. HOMOGENEOUS MATERIAL PARAMETERS

Hyperbolic layer stripping procedures are well-conditioned problems. This is because a point in space-time, for one of the coupled one-way wave equations has only one line of dependency. Therefore there is a simple travel time map enabling layer stripping type algorithms to proceed. This is *not* true for parabolic problems. These problems have solutions of non-compact support even when the initial condition has compact support. This is a manifestation of the *infinite speed of propagation* synonymous with these equations.

We start by examining the solution properties for the inverse problem associated with parabolic and hyperbolic versions of the problem when the medium parameters are constant.

**3.1. Parabolic Advection equation with constant coefficients.** Consider the equation (2.4) with  $v \equiv 0$ ,  $r \equiv 0$  on the semi-infinite  $x$ -axis and with all material parameters homogeneous

$$(3.1) \quad \kappa^{-2} \bar{v} \partial_x c + \kappa^{-2} \partial_t c = \partial_x^2 c, \quad 0 < x < \infty, \quad t > 0,$$

with prescribed initial-boundary values

$$(3.2) \quad \left. \begin{aligned} c(0, t) &= h(t), \quad t > 0, \\ c(x, 0) &= 0, \quad 0 < x < \infty. \end{aligned} \right\}$$

Equation (3.1) is parabolic and therefore requires two boundary conditions for well-posedness; boundedness of the solution as  $x \rightarrow \infty$  is also imposed as well as the boundary value at  $x = 0$ .

The inverse problem we pose is one of measuring the concentration at a station  $x = \ell$ , i.e.,

$$(3.3) \quad c(\ell, t) = f(t),$$

and from this it is required to estimate the boundary function  $h(t) = c(0, t)$ . The direct map is described through (3.1).

It can be shown that the solution to this initial-boundary value problem is

$$(3.4) \quad c(x, t) = \int_0^t k(x, s) h(t - s) ds,$$

where the kernel  $k(x, t)$  is

$$(3.5) \quad k(x, t) = \frac{x}{2\kappa\sqrt{\pi t^{3/2}}} \exp\left(\frac{-(x - \bar{v}t)^2}{4\kappa^2 t}\right).$$

Since (3.5) defines a delta family (Stakgold, 1979), (p. 110), it follows that  $\lim_{x \rightarrow 0} k(x, t) = \delta(t)$ , where  $\delta$  is the Dirac delta distribution, and  $k(x, t)$  is the kernel in (3.5). When  $\kappa \rightarrow \infty$ , which corresponds to perfect mixing where the fluids are instantaneously mixed,  $\lim_{\kappa \rightarrow \infty} k(x, t) = \delta(t)$ , and the solution to (3.4) is

$$(3.6) \quad \lim_{\kappa \rightarrow \infty} c(x, t) = h(t).$$

When  $\kappa \rightarrow 0$ , the flow corresponds to perfect displacement and diffusion disappears. It follows that  $\lim_{\kappa \rightarrow 0} k(x, t) = \delta(x - \bar{v}t)$ , and the solution to equation (3.4) then reduces to

$$(3.7) \quad \lim_{\kappa \rightarrow 0} c(x, t) = h(t - \bar{v}^{-1}x) H(t - \bar{v}^{-1}x).$$

This is intuitively correct, as in the limit  $\kappa \rightarrow 0$ , the equation (3.1) becomes the advective equation. The final limit to consider is when  $\bar{v} \rightarrow \infty$ , where the advective effects dominate the diffusive effects. In this case  $\lim_{\bar{v} \rightarrow \infty} k(x, t) = \delta(t)$ , and the solution to (3.4) is given by

$$(3.8) \quad \lim_{\bar{v} \rightarrow \infty} c(x, t) = h(t).$$

For non-zero  $\kappa$  the kernel provides a non-localised propagation mechanism in contra-distinction to the case when  $\kappa = 0$ , where the equation (3.1) is hyperbolic. In this limit case the initial condition will propagate along the characteristics  $t = \bar{v}^{-1}x$ . In order to perform layer stripping it is necessary that the physical phenomenon which is used to probe the medium propagates at finite

speed and has a wave-like behaviour. The solution for (3.1) has such behaviour when  $\kappa$  is very small. If  $\kappa$  is moderately large, it is not possible to solve the problem by layer stripping, see (I) and (Vogel, 1992). One feature of the solution to (3.1), is that a wave will propagate at a velocity near  $\bar{v}$ , but only in the direction of increasing  $x$ . This means the equation (3.1) can be considered as a one-way wave equation; as such no reflection experiments are possible. Transmission measurements are the only possible method of solving the inverse problem; see (Wall & Lundstedt, 1998) for further information on inverse problems for a one-way wave equation.

The regularisation of the inverse problem of source concentration reconstruction by mollification has been examined in (Shorten, 2000; Shorten & Wall, 2001). We now consider regularisation of the problem by hyperbolisation of the parabolic problem.

**3.2. Hyperbolic Advection Equation.** The use of a hyperbolic problem to regularise parabolic problems seems a sensible physical extension, given the development of §2. We show here that the equivalent hyperbolic inverse problem to that discussed in §3.23.1 is well conditioned.

When the material parameters are homogeneous equation (2.4) becomes

$$(3.9) \quad v^2 \partial_t^2 c + v^2 \bar{v} \partial_t \partial_x c + \kappa^{-2} \bar{v} \partial_x c + \kappa^{-2} \partial_t c = \partial_x^2 c, \quad 0 < x < \infty, \quad t > 0,$$

with prescribed initial-boundary values

$$(3.10) \quad \left. \begin{aligned} c(0, t) &= h(t) \quad t > 0, \\ c(x, 0) &= 0, \quad 0 < x < \infty. \end{aligned} \right\}$$

Equation (3.9) is hyperbolic and therefore requires two boundary conditions for well-posedness; boundedness of the solution as  $x \rightarrow \infty$  is also imposed as well as the boundary value at  $x = 0$ . It is seen that as  $v \rightarrow 0$ , (3.1) can be considered a singular perturbation of (3.9).

To examine the well-posedness of the problem we extend the function  $c(x, t)$ , and the partial differential equation (3.9), by zero for  $t < 0$ , and consider its Fourier transform. The Fourier transform of the dependent variable is pivotal in our argument; it is

$$(3.11) \quad \hat{c}(x, \xi) = \hat{h}(\xi) \exp\left(\frac{\kappa^{-2} \bar{v}}{2} x\right) \exp\left(i \xi \frac{v^2 \bar{v}}{2} x - \sqrt{\frac{\beta |\xi|}{2 \kappa^2}} I(\xi, a) x\right),$$

with

$$(3.12) \quad a = \frac{1}{\beta} \left( \frac{\bar{v}^2}{4 \xi \kappa^2} - \xi v^2 \kappa^2 \tilde{\beta} \right), \quad \beta = 1 + \frac{v^2 \bar{v}^2}{2}, \quad \tilde{\beta} = 1 + \frac{v^2 \bar{v}^2}{4},$$

and where

$$(3.13) \quad I(\xi, a) = (\sqrt{1 + a^2} + \sigma a)^{1/2} + i \sigma (\sqrt{1 + a^2} - \sigma a)^{1/2},$$

with  $\sigma = \text{sign}(\xi)$ <sup>4</sup>. When use is made of the boundary condition (3.10) it can be seen that for the inverse signal reconstruction problem, the operator mapping  $f \rightarrow h$  in the Fourier transform

<sup>4</sup>Both references (Murio, 1993) and (Murio & Roth, 1988) have typographical errors in the definition of a function similar to our  $I$  function.



domain can be written as

$$(3.14) \quad \widehat{h}(\xi) = \widehat{f}(\xi) \exp\left(-\frac{\kappa^{-2}\bar{v}}{2}\ell\right) \exp\left(-i\xi \frac{v^2\bar{v}}{2}\ell + \sqrt{\frac{\beta|\xi|}{2\kappa^2}} I(\xi, a)\ell\right).$$

It should be noted, forgetting any ill-posedness, that the function  $h$ , denoted here by  $h^H$  reconstructed from (3.14), will not be equal to the solution reconstructed from (3.1), denoted here by  $h^P$ , as the later is associated with the parabolic problem, while the former is associated with the hyperbolic problem. This is just a statement that the inverse mapping operators for these two problems are not identical, although as  $v \rightarrow 0$ ,  $\|h^H - h^P\| \rightarrow 0$  with a rate of convergence given by the following consistency estimate.

**Lemma 3.1.** *If  $h^P \in H^1$  and  $M = \max(\|h^P\|, \|h^{P'}\|)$ , then  $\|h^H - h^P\| = M\mathcal{O}(v^2)$ .*

*Proof.* Using (3.14), and (3.11) when  $v = 0$ , it follows that

$$(3.15) \quad \|h^P - h^H\|_2 = \|(1 - \chi)\widehat{h^P}(\xi)\|_2.$$

where

$$(3.16) \quad \chi = \exp\left(\frac{-i\xi v^2\bar{v}\ell}{2} + \sqrt{\frac{|\xi|}{2\kappa^2}} \left(\sqrt{\beta}I(\xi, a) - I\left(\xi, \frac{\bar{v}^2}{4\xi\kappa^2}\right)\right)\ell\right),$$

with  $0 \leq |\chi| \leq 1$ , and  $\lim_{v \rightarrow 0} |\chi| = 1$ . Since  $h^P \in H^1$ , then  $\xi\widehat{h^P}(\xi) \in L^2$ , and it follows that  $\exists N > 0$  such that

$$(3.17) \quad \|h^P - h^H\|_2 = \sqrt{4 \int_0^N |1 - \chi|^2 |\widehat{h^P}|^2 d\xi}$$

$$(3.18) \quad \leq 2M\sqrt{N} \max_{\xi \in [0, N]} |1 - \chi|$$

$$(3.19) \quad \leq 2M\sqrt{N}, \quad \forall v$$

This bounds the error for large  $v$ . For small  $v$ , a series expansion of (3.18) about  $v = 0$  yields

$$(3.20) \quad \|h^P - h^H\|_2 = M\mathcal{O}(v^2), \quad \forall v.$$

A weaker form of this lemma when  $\bar{v} = 0$  appears in (Murio & Roth, 1988; Murio, 1993; Roth, 1989).  $\square$

The important behaviour of (3.14) is as  $|\xi| \rightarrow \infty$  and we observe that

$$(3.21) \quad \max_{|\xi| \in [0, \infty]} \left| \sqrt{\frac{\beta|\xi|}{2\kappa^2}} \mathcal{R}e(I(\xi, a))\ell \right| \leq \max\left\{ \frac{\bar{v}\ell}{2\kappa^2}, \frac{\beta\ell}{2\kappa^2 v \bar{\beta}^{\frac{1}{2}}} \right\} = \bar{k}.$$

It is now seen that the effect of hyperbolicity is to bound the growth of the exponential function for large values of  $|\xi|$ , with

$$(3.22) \quad \left| \exp\left(\sqrt{\frac{\beta|\xi|}{2\kappa^2}} \mathcal{R}e(I(\xi, a))\ell\right) \right| < e^{\bar{k}}.$$

It therefore follows that

$$(3.23) \quad \|\widehat{h^H} - \widehat{h_m^H}\|_2^2 \leq e^{\bar{k}} \|\widehat{f_m} - \widehat{f}\|_2^2,$$

where  $h_m$  corresponds to  $u(0, t)$  when  $f$  is replaced by  $f_m$ . The following stability result then can be obtained from Parseval's theorem.

**Lemma 3.2.** *If  $f, f_m \in L^2$  then*

$$(3.24) \quad \|h^H - h_m^H\|_2^2 \leq e^{\bar{k}} \|f_m - f\|_2^2$$

We see the hyperbolic problem ensures that the inverse mapping operator has a Lipschitz continuity result, when the data  $f_m \in C$ , and  $v > 0$  is fixed. Furthermore, considering the hyperbolic problem (3.9) as a regularisation of (3.1), as  $\|f_m - f\| \rightarrow 0$ ,  $v$  can also be reduced. Then well-posedness of the inverse problem follows from the two lemmata.

**Theorem 3.3.** *The inverse problem is stable with respect to perturbations in the data  $f$ . If the exact boundary function  $h^P \in C^1$  with  $\max(|h^P|, |h^{P'}|) < M$  then the solution  $h^P$  to the inverse problem satisfies*

$$(3.25) \quad \|h^P - h_m^H\|_2^2 \leq e^{\bar{k}} \|f - f_m\|_2^2 + \mathcal{O}(v^2)$$

The well-posedness of the problem is also apparent from the solution to (3.9) which is

$$(3.26) \quad c(x, t) = \exp\left(\frac{\bar{v}x}{2\kappa^2}\left(1 - \frac{\beta}{2\bar{\beta}}\right)\right) \left[ \exp\left(\frac{-\beta x}{2\kappa\sqrt{\tau}\sqrt{\bar{\beta}}\gamma_+}\right) h\left(t - \frac{xv}{\gamma_+}\right) + \int_{\frac{xv}{\gamma_+}}^t k(x, s) h(t-s) ds \right],$$

where the kernel is

$$(3.27) \quad k(x, s) = \exp\left(\frac{-\beta s}{2\tau\sqrt{\bar{\beta}}}\right) \left( \frac{xv\sqrt{\bar{\beta}}\alpha I_1\left(\alpha\sqrt{\left(s - \frac{xv}{\gamma_+}\right)^2 + 2xv\sqrt{\bar{\beta}}\left(s - \frac{xv}{\gamma_+}\right)}\right)}{\sqrt{\left(s - \frac{xv}{\gamma_+}\right)^2 + 2xv\sqrt{\bar{\beta}}\left(s - \frac{xv}{\gamma_+}\right)}} \right),$$

with boundary condition  $h(t) \equiv 0, t < 0$ , where  $I_n$  denotes the modified Bessel function of order  $n$ , and

$$(3.28) \quad \alpha = \frac{1}{2\tau\bar{\beta}}\sqrt{\beta^2 - \bar{\beta}\bar{v}^2v^2}, \quad \text{and} \quad \gamma_+ = \sqrt{\bar{\beta}} + \sqrt{\bar{\beta} - 1}.$$

Therefore the inverse problem with homogeneous material parameters can be formulated as a Volterra integral equation of the second kind, and the inversion of this equation provides the solution to the signal restoration problem. The theory of second kind Volterra operators implies that the problem is well-posed (Linz, 1985).

The first part of the solution on the right-hand-side of (3.26) represents the hyperbolic wave that travels into the medium undistorted but with attenuation. From this part of the solution it

can be seen, that the distance into the medium, in which the leading edge of the wave traveling twice this distance is attenuated by  $e^{-1}$ , the so-called e-fold distance (Weston, 1988), is

$$(3.29) \quad x_e = \kappa \sqrt{\tau} \beta^{-1} \sqrt{\tilde{\beta} \gamma_+}.$$

The maximum e-fold distance occurs when  $v\bar{v} = \sqrt{2(\sqrt{2}-1)}$ , and  $\kappa\sqrt{\tau} \leq x_e < 1.21\kappa\sqrt{\tau}$ . The second part of the solution, represented by the convolution integral, directly represents the dissipative or diffusive nature of the problem. Further discussion on the interpretation of this equation can be found in (I).

If the term  $v^2\bar{v}\partial_t\partial_x c$  is omitted from (3.9), then the hyperbolic partial differential equation

$$(3.30) \quad v^2\partial_t^2 c + \kappa^{-2}\bar{v}\partial_x c + \kappa^{-2}\partial_t c = \partial_x^2 c, \quad 0 < x < \infty, \quad t > 0,$$

can still be considered a valid singular perturbation of the parabolic problem (3.1). The solution for the concentration field  $c$  within a semi-infinite region with zero initial condition can be found by Laplace transform techniques to be

$$(3.31) \quad c(x, t) = \exp\left(\frac{\bar{v}x}{2\kappa^2}\right) \left[ \exp\left(\frac{-x}{2\kappa\sqrt{\tau}}\right) h(t - vx) + \int_{vx}^t k(x, s) h(t - s) ds \right],$$

where the kernel of the integral is

$$k(x, t) = \exp\left(-\frac{t}{2\tau}\right) \frac{\alpha vx I_1(\alpha \sqrt{t^2 - v^2 x^2})}{\sqrt{t^2 - v^2 x^2}},$$

with boundary condition  $h(t) \equiv 0, t < 0$ , and

$$\alpha = \frac{1}{2\tau} \sqrt{1 - \bar{v}^2 v^2}.$$

Note that when  $\bar{v} = 0$ , the solutions in (3.26) and (3.31) are identical to the solution found in (I). Again the first part of the solution on the right-hand-side of (3.31) represents the hyperbolic wave that travels into the medium undistorted but with attenuation, and the second part of the solution is associated with the problem dispersion. In this case the e-fold distance is  $x_e = \kappa\sqrt{\tau}$ .

For the inverse problem under consideration the measurement is  $c(\ell, t)$ , so that equation (3.31) can be written as

$$(3.32) \quad c(\ell, t) = \exp\left(\frac{\bar{v}\ell}{2\kappa^2}\right) \left[ \exp\left(\frac{-\ell}{2\kappa\sqrt{\tau}}\right) h(t - v\ell) + \int_{v\ell}^t k(\ell, s) h(t - s) ds \right],$$

and again the inverse problem with homogeneous material parameters can be formulated as a Volterra integral equation of the second kind, with the inversion of this equation providing the well-posed solution to the signal restoration problem. This integral equation approach has been examined in (Murio & Roth, 1988) for the parabolic heat equation.

## 4. WAVE SPLITTING AND SYSTEM DYNAMICS

We now return to obtaining numerically useful techniques based on wave splitting ideas to find algorithms for the inverse problem when the material parameters are spatially varying.

The equations (2.3) and (2.4), with  $r \equiv 0$ , can now both be written in the system form

$$(4.1) \quad \partial_x \mathbf{u} = \mathbf{C}\mathbf{u} + \mathbf{B}\mathbf{u},$$

where for the conversion of the second order partial differential equations  $\mathbf{u} = [c \quad \partial_x c]^T$ , with component matrices

$$(4.2) \quad \mathbf{C} = \begin{bmatrix} 0 & 1 \\ (v^2 \partial_t^2 + \kappa^{-2} \partial_t) & v^2 \bar{v} \partial_t \end{bmatrix},$$

$$\mathbf{B} = \begin{bmatrix} 0 & 0 \\ \partial_x(\bar{v})(\kappa^{-2} + v^2 \partial_t) & \kappa^{-2} \bar{v} - 2\partial_x(\ln(\kappa)) \end{bmatrix}.$$

In the system case, given by (2.3),  $\mathbf{u} = [c \quad J]^T$  and the component matrices are

$$(4.3) \quad \mathbf{C} = \begin{bmatrix} 0 & -\kappa^{-2}(\tau \partial_t + 1) \\ -\partial_t & \bar{v} \kappa^{-2} \tau \partial_t \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 0 & 0 \\ -\partial_x(\bar{v}) & \bar{v} \kappa^{-2} \end{bmatrix}.$$

It should be noted that the preliminary partitions<sup>5</sup>, of the matrices  $\mathbf{C}$  and  $\mathbf{B}$ , used in the previous equations are not unique and other partitions may be profitable. The particular choice we have made includes the advection velocity and this seems to be essential in order to provide the methods developed in the next sections. In the next section we diagonalise the operator matrix  $\mathbf{C}$ ; the  $\mathbf{B}$  matrix contains only terms irrelevant to this. A consistent partition has been chosen for  $\mathbf{C}$  in equations (4.2) and (4.3) to ensure that they are *similar*, namely they have the same eigenvalue operator-valued matrices (see (4.22)).

We note that the  $\mathbf{C}$  matrix has an extra advection derivative term, the term  $v^2 \bar{v} \partial_t$  for (4.2), and the term  $\bar{v} \kappa^{-2} \tau \partial_t$  for (4.3), when compared to the wave splitting for one dimensional hyperbolic diffusion wave equations of (I). We have included this term in  $\mathbf{C}$  because any physically realistic media involving mass transport must involve advection. To not include this term in the  $\mathbf{C}$  matrix will result in non-physical split fields, which will mean that we will be unable to measure or reconstruct the appropriate field to solve the inverse problem specified in §3.3.1. One central feature of this inverse problem investigation is that we assume that either the reconstruction or measurements (or both) are carried out in an advective diffusive medium — not an ideal non-diffusive medium. An alternative splitting to the one discussed here is mentioned in A.

We shall make use of diagonalising transformations to convert the equations (4.1) into the appropriate split form. The pseudo-differential operators found in this section are most easily found by using Laplace transformation techniques on the  $\mathbf{C}$  matrices given in equations (4.2), (4.3), and then finding the algebraic eigen-systems. The pseudo-differential operators are then found by inverse Laplace transformation. Examples of this technique are found in (I).

<sup>5</sup>We use the term preliminary, as a splitting based on this partition will be used later to diagonalise the operator matrix  $\mathbf{C}$ .

We illustrate the splitting for the second order equation formulation only; the splitting operators for the system representation are similar.

**4.1. Second Order equation.** Using Laplace transform techniques, it is found that the appropriate diagonalising transformation for the  $\mathbf{C}$  matrix (4.2), is given by

$$(4.4) \quad \mathbf{u} = \mathbf{P}\mathbf{v}^\pm.$$

The  $\mathbf{v}^\pm$  have the properties of right and left moving waves and this is discussed further latter in this section. Note that it is assumed that

$$(4.5) \quad c(x, 0) = \partial_t c(x, t)|_{t=0} = 0.$$

The operator-valued matrix in the equation (4.4) is defined by

$$\mathbf{P} = \begin{bmatrix} \mathbb{I} & \mathbb{I} \\ \kappa^{-1}\mathbb{K}_-^{-1} & \kappa^{-1}\mathbb{K}_+^{-1} \end{bmatrix},$$

where  $\mathbb{I}$  is the identity operator. The inverse operator  $\mathbb{K}_\pm^{-1}$  is the operator representation of the temporal pseudo-differential operator defined through

$$(4.6) \quad \mathbb{K}_\pm^{-1} = \left( \gamma\sqrt{\tau}\partial_t \pm \sqrt{\partial_t(\chi\partial_t + 1)} \right),$$

where the ratio of the advection velocity to the hyperbolic wavespeed is

$$\gamma = \frac{\bar{v}\mathbf{v}}{2} = \frac{\bar{v}\kappa^{-1}\sqrt{\tau}}{2},$$

and

$$\chi = \tau(1 + \gamma^2).$$

With the definition

$$\gamma_\pm = (\gamma \pm \sqrt{1 + \gamma^2}),$$

the operator  $\mathbb{K}_\pm^{-1}$  has the representation

$$(4.7) \quad \mathbb{K}_\pm^{-1}f = \left( \gamma_\pm\sqrt{\tau}\partial_t \pm \frac{1}{2\sqrt{\chi}}(\mathbb{I} - \mathbb{L}) \right)f(t),$$

where  $\mathbb{L}$  is the convolution operator

$$(4.8) \quad (\mathbb{L}f)(t) = \int_0^t L(x, t-t')f(t') dt',$$

with kernel

$$(4.9) \quad L(x, t) = \exp\left(\frac{-t}{2\chi}\right) \frac{\mathbb{I}_1(t/2\chi)}{t}.$$

In this equation  $I_n$  denotes the modified Bessel function of order  $n$ . The operator  $\mathbb{K}_\pm$ , which is central in the operator-valued matrix  $\mathbf{P}^{-1}$ , is the pseudo-differential operator defined through

$$(4.10) \quad \mathbb{K}_\pm = 1/[\gamma\sqrt{\tau}\partial_t \pm \sqrt{\partial_t(\chi\partial_t + 1)}],$$

and this operator can be represented by the convolution operator

$$(\mathbb{K}_\pm f)(t) = \int_0^t \tilde{K}_\pm(x, t-t')f(t') dt'.$$

A closed form representation for the kernel  $\tilde{K}_\pm$  can be obtained, although the kernel involves a convolution term. In this paper it suffices to find a convergent series representation for  $\tilde{K}_\pm$ , which for  $\gamma < 1$  is

$$(4.11) \quad \tilde{K}_\pm(x, t) = \pm K_\tau(x, t) - \frac{\gamma}{\sqrt{\tau}} e^{-t/\tau} \pm \frac{\gamma^2}{2\sqrt{\tau}} \left( F\left(\frac{3}{2}, 1, -\frac{t}{\tau}\right) - \frac{\gamma^2}{4} F\left(\frac{5}{2}, 1, -\frac{t}{\tau}\right) + \frac{\gamma^4}{8} F\left(\frac{7}{2}, 1, -\frac{t}{\tau}\right) + \mathcal{O}(\gamma^6) \right),$$

where the kernel  $K_\tau$  is the same kernel found for the corresponding operator for thermal processes in (I), namely

$$(4.12) \quad K_\tau(x, t) = \frac{1}{\sqrt{\tau}} e^{-t/2\tau} I_0(t/2\tau).$$

The convolution operator corresponding to this kernel is

$$(4.13) \quad (\mathbb{K}_\tau f)(t) = \int_0^t K_\tau(x, t-t')f(t') dt'.$$

The term  $F(n, d, z)$ , appearing in (4.11), is the generalised hypergeometric function. This function is also known as the KummerM function (Abramowitz & Stegun, 1964), (p. 504), and is represented by the absolutely convergent series

$$F(n, d, z) = \sum_{k=0}^{\infty} \frac{\Gamma(n+k)\Gamma(d)z^k}{\Gamma(n)\Gamma(d+k)k!},$$

where  $\Gamma$  is the gamma function.

The inverse of the operator-valued matrix  $\mathbf{P}$  is then found to be

$$(4.14) \quad \mathbf{P}^{-1} = \frac{1}{2} \begin{bmatrix} \mathbb{I} + \frac{\gamma}{\sqrt{1+\gamma^2}}(\mathbb{M} + \mathbb{I}) & -\kappa\mathbb{K}_\chi \\ \mathbb{I} - \frac{\gamma}{\sqrt{1+\gamma^2}}(\mathbb{M} + \mathbb{I}) & \kappa\mathbb{K}_\chi \end{bmatrix},$$

with the operator  $\mathbb{M}$  having the representation

$$(4.15) \quad \mathbb{M}f = \int_0^t M(x, t-t')f(t') dt',$$

with kernel  $M$  given by

$$M(x, t) = \frac{e^{-t/2\chi}}{2\chi} \left( I_1(t/2\chi) - I_0(t/2\chi) \right).$$

The operator  $\mathbb{K}_\chi$  is defined by the same representation as  $\mathbb{K}_\tau$ , namely (4.13), but the variable  $\tau$  in the kernel of this operator, given by equation (4.12), is replaced by  $\chi$ . It is to be noted that  $\partial_t K_\chi = \sqrt{\chi} M$ .

**4.2. Operator Properties.** We proceed formally, and list some of the algebraic properties of the operators developed in the last section. These relationships may be proved via direct manipulation of the operator representations, or through the Laplace transform. As the operators  $\mathbb{K}_\pm^{-1}$  are roots of a quadratic characteristic equation they satisfy certain composition, commutation, and trace properties. These are:

$$(4.16) \quad \mathbb{K}_-^{-1} \mathbb{K}_+^{-1} = \mathbb{K}_+^{-1} \mathbb{K}_-^{-1} = \mathcal{L}_t \equiv -\partial_t(\tau \partial_t + 1)$$

$$(4.17) \quad \mathbb{K}_\pm^{-1} = \mathcal{L}_t \mathbb{K}_\mp = \mathbb{K}_\mp \mathcal{L}_t = \partial_t \mathbb{K}_\mp(\tau \partial_t + 1)$$

$$\mathbb{K}_\pm^{-1} \mathbb{K}_\pm = \mathbb{K}_\pm \mathbb{K}_\pm^{-1} = \mathbb{I}$$

$$(4.18) \quad \begin{aligned} \mathbb{K}_+^{-1} + \mathbb{K}_-^{-1} &= 2\gamma\sqrt{\tau}\partial_t \\ \mathbb{K}_+^{-1} - \mathbb{K}_-^{-1} &= 2\sqrt{\partial_t(\chi\partial_t + 1)} \end{aligned}$$

Relations (4.16) and (A) define the operators  $\mathbb{K}_\pm^{-1}$ , and if the right-hand-side of (A) is zero these relations provide the definition of the square root of  $-\mathcal{L}_t$  (c.f. §3 of (I)). The relationship (4.17) is often called the commutation relation, it appears in many wave splitting problems, albeit with operators different to those considered here. These equations are background to many of the results of this paper.

The operators  $\mathbb{K}_\pm$  are smoothing, compact on the Hilbert space  $L_2$ , and as such the inverse operators  $\mathbb{K}_\pm^{-1}$  are unbounded and ill-posed on  $L_2$ , even though existence of the operators has been proven by construction. We only consider the mapping properties of the operators when  $\bar{\nu} \neq 0$ , as the case for  $\bar{\nu} \equiv 0$  has been covered in (I).

**Theorem 4.1.** *For  $\tau \neq 0$ , the operators are injective and into on the appropriate Sobolev spaces*

$$\begin{aligned} \mathbb{K}_\pm : H^s &\mapsto H^{s+1}, \\ \mathbb{K}_\pm^{-1} : H^s &\mapsto H^{s-1}, \end{aligned}$$

where the Sobolev space of order  $m$  is denoted by  $H^m$ .

*Proof.* To prove the operators are injective it is only necessary to look at the image of the zero function because the operators are linear; it follows trivially from their explicit form they are injective. The mapping properties of the operators follows directly from their Laplace transforms and the symbol mapping theorem (Taylor, 1981), (p. 49 et seq.).  $\square$

In the limiting case of pure diffusion  $\bar{v} \rightarrow 0$ ,  $\gamma \rightarrow 0$  and  $\chi \rightarrow \tau$ , so it follows that the operators reduce to those of (I):

$$\mathbb{K}_{\pm}^{-1} \rightarrow \pm \mathbb{K}_{\tau}^{-1}, \quad \mathbb{K}_{\pm} \rightarrow \pm \mathbb{K}_{\tau}.$$

Of major concern also, is the limiting forms of the operators  $\mathbb{K}_{\pm}$  as  $\tau \rightarrow 0$ ; when  $\tau = 0$ , the model equations are parabolic. Define

$$\mathbb{H}f = \int_0^t H(t-s)f(s)ds,$$

with kernel  $H(t) = 1/\sqrt{\pi t}$ , then  $\mathbb{H}f$  is related to the half derivative of  $f$ , that is  $\mathbb{H}f = \partial_t^{-1/2}f$ . The half derivative has the obvious composition properties (Oldham & Spanier, 1974)

$$\partial_t^{1/2}f = \partial_t \mathbb{H}f = \partial_t \partial_t^{-1/2}f.$$

It is then possible to show

$$\lim_{\tau \rightarrow 0} \mathbb{K}_{\pm} = \pm \mathbb{H},$$

and when representation (4.7) is used for  $\mathbb{K}_{\pm}^{-1}$

$$\lim_{\tau \rightarrow 0} \mathbb{K}_{\pm}^{-1} = \pm \partial_t \mathbb{H}.$$

Finally the limit as the equation becomes non-diffusive can be achieved by considering the limit as  $\kappa^{-1} \rightarrow 0$  while keeping  $v$  fixed, or equivalently allowing  $\tau \rightarrow \infty$ , again while keeping  $v$  fixed. It can be shown that

$$(4.19) \quad \lim_{\tau \rightarrow \infty} \kappa \mathbb{K}_{\pm} = \frac{1}{v\gamma_{\pm}} \partial_t^{-1}, \quad \lim_{\tau \rightarrow \infty} \kappa^{-1} \mathbb{K}_{\pm}^{-1} = v\gamma_{\pm} \partial_t,$$

which is the corresponding splitting for the hyperbolic partial differential equation

$$(4.20) \quad v^2 \partial_t^2 c + v^2 \bar{v}^2 \partial_t \partial_x c = \partial_x^2 c.$$

In the limit as  $\bar{v} \rightarrow 0$ ,  $\gamma \rightarrow 0$ , and  $\gamma_{\pm} \rightarrow \pm 1$ , with (4.19) reducing to the standard splitting for the wave equation. Observe that  $\gamma_- < 0$  for  $\gamma > 0$ <sup>6</sup>, and as  $\gamma \rightarrow \infty$ ,  $\gamma_- \rightarrow 0$  and  $\gamma_+ \rightarrow 2\gamma$ . Figure 1 shows how the wave phase velocity varies with the advection velocity. The relationship between the phase velocity  $\gamma_+$ , and the advection velocity is only approximately linear for  $\frac{\bar{v}v}{2} > 1$ . This nonlinear behaviour is expected, as the velocity of the propagation of the mass wave through the advecting medium is not associated with a Galilean transformation.

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<sup>6</sup>Note also that  $\gamma_+ \gamma_- = -1$ .



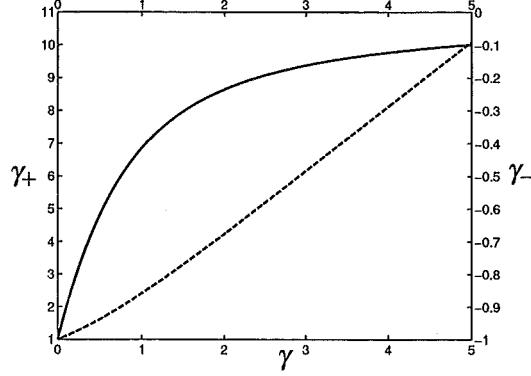


FIGURE 1. The variation of the wavespeed parameters,  $\gamma_+$  (---) and  $\gamma_-$  (—) with  $\gamma$ .

**4.3. System Dynamics.** The transformation (4.4) can now be utilised to diagonalise  $\mathbf{C}$ , and to convert equation (4.1) into two coupled one-way wave equations

$$(4.21) \quad \partial_x \mathbf{v}^\pm = \Lambda \mathbf{v}^\pm + \mathbf{D} \mathbf{v}^\pm,$$

with the new basis  $\mathbf{v}^\pm = [v^+ \ v^-]^T$ , where  $\{v^+, v^-\}$  have the properties of right and left moving mass concentration waves; this is discussed further latter in this section. The matrix  $\Lambda$  is the diagonal operator matrix

$$(4.22) \quad \Lambda = \begin{bmatrix} \kappa^{-1} \mathbb{K}_-^{-1} & 0 \\ 0 & \kappa^{-1} \mathbb{K}_+^{-1} \end{bmatrix},$$

and the dynamics matrix  $\mathbf{D}$  is

$$(4.23) \quad \mathbf{D} = -\mathbf{P}^{-1}(\partial_x \mathbf{P}) + \mathbf{P}^{-1} \mathbf{B} \mathbf{P}.$$

It is important to note that the diagonal matrix  $\Lambda$  given by (4.22) will be the same for system (2.3) and the second order equation (2.4), so that the principal part of the dynamics equation will be the same for both these equations. In the sequel we only quote results for the second order equation as analysed in §4.4.1. Results for the system equations in (4.3) are of a similar nature, but have different functional forms from those listed in Table 1 and Table 2, and will *not* be listed here.

In terms of the material parameters the first part of the system dynamics is

$$(4.24) \quad \mathbf{P}^{-1}(\partial_x \mathbf{P}) = \begin{bmatrix} -1 & -1 \\ 1 & 1 \end{bmatrix} \frac{j(x)}{\sqrt{1+\gamma^2}} (\mathbb{M} + \mathbb{I}) + \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} d(x) (\mathbb{I} + e(x) \mathbb{J}),$$

with  $\mathbb{J}$  represented by the convolution operator

$$(4.25) \quad (\mathbb{J}f)(t) = \int_0^t J(t-t') f(t') dt',$$

Coefficient	Second order equations (2.4)
$d(x)$	$\frac{1}{4}\partial_x \ln[v^2(1+\gamma^2)]$
$e(x)$	$-\frac{1}{\tau(1+\gamma^2)} + \frac{\partial_x[\kappa^{-2}]}{\partial_x[v^2(1+\gamma^2)]}$
$f(x)$	$\frac{\gamma}{2}(\kappa^{-2}\bar{v} - 2\partial_x(\ln \kappa)) + (\partial_x \bar{v})\kappa^{-1}\sqrt{\tau}$
$g(x)$	$\frac{1}{2}(\kappa^{-2}\bar{v} - 2\partial_x \ln \kappa)$
$h(x)$	$\frac{1}{2\kappa}\partial_x \bar{v}$
$j(x)$	$\frac{1}{2\bar{v}}\partial_x(v\gamma)$

TABLE 1. Identification of parameters  $d - j$  for hyperbolic mass transport in the second order equation case.

with kernel  $J(t) = \exp(-t/\chi)$ . The spatial functions in equation (4.24) are shown in Table 1, and the remaining part of the dynamics, after transformation, is

$$(4.26) \quad \mathbf{P}^{-1}\mathbf{B}\mathbf{P} = \begin{bmatrix} -1 & -1 \\ 1 & 1 \end{bmatrix} \left( h(x)\mathbb{K}_\chi + \frac{f(x)}{\sqrt{1+\gamma^2}}(\mathbb{M} + \mathbb{I}) \right) + \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} g(x)\mathbb{I},$$

with the coefficients  $d, e, f, g, h$ , and  $j$  given in Table 1.

For explicitness we express the system (4.21) in terms of the dynamics matrix as

$$(4.27) \quad \partial_x \mathbf{v}^\pm(x, t) = \Lambda \mathbf{v}^\pm(x, t) + \begin{bmatrix} \alpha & \beta \\ \tilde{\gamma} & \delta \end{bmatrix} \mathbf{v}^\pm(x, t) + \int_0^t \begin{bmatrix} A(x, t-t') & B(x, t-t') \\ C(x, t-t') & D(x, t-t') \end{bmatrix} \mathbf{v}^\pm(x, t') dt',$$

(c.f. (Åberg, Kristensson, & Wall, 1995) for a similar case where the dynamics include operators<sup>7</sup>). The first part of the dynamics represented by the terms  $\{\alpha, \beta, \tilde{\gamma}, \delta\}$  are listed in Table 2, and these terms are purely multiplicative functions. The part of the dynamics corresponding to integral operators has been split into the convolutional term; the kernels of these operators are also listed in Table 2. The convolutional operators corresponding to the kernels  $\{A, B, C, D\}$  will be denoted by  $\{\mathbb{A}, \mathbb{B}, \mathbb{C}, \mathbb{D}\}$ , respectively. It is to be observed that it is necessary to split the dynamics into functions and operators in order to derive the Green operator equations of the next section.

<sup>7</sup>We note that the equations for the reflection kernel, and the Green operators derived in (I) are derived under the assumption that the lateral loss term, there denoted by  $\chi$ , was  $\chi \equiv 0$ , and the relaxation time  $\tau \neq \tau(x)$ ; this was *not* specified in the cited paper.

Coefficient	Terms of material parameters
$\alpha(x)$	$-\frac{1}{\sqrt{1+\gamma^2}}(f(x) - j(x)) + (g(x) - d(x))$
$\beta(x)$	$-\frac{1}{\sqrt{1+\gamma^2}}(f(x) - j(x)) - (g(x) - d(x))$
$\tilde{\gamma}(x)$	$\frac{1}{\sqrt{1+\gamma^2}}(f(x) - j(x)) - (g(x) - d(x))$
$\delta(x)$	$\frac{1}{\sqrt{1+\gamma^2}}(f(x) - j(x)) + (g(x) - d(x))$
Kernel	Terms of material parameters
$A(x, t)$	$-\frac{1}{\sqrt{1+\gamma^2}}((f(x) - j(x))M(x, t) - h(x)K(x, t) - d(x)e(x)J(x, t))$
$B(x, t)$	$-\frac{1}{\sqrt{1+\gamma^2}}((f(x) - j(x))M(x, t) - h(x)K(x, t) + d(x)e(x)J(x, t))$
$C(x, t)$	$\frac{1}{\sqrt{1+\gamma^2}}((f(x) - j(x))M(x, t) + h(x)K(x, t) + d(x)e(x)J(x, t))$
$D(x, t)$	$\frac{1}{\sqrt{1+\gamma^2}}((f(x) - j(x))M(x, t) + h(x)K(x, t) - d(x)e(x)J(x, t))$

TABLE 2. Identification of dynamics coefficients and kernels for hyperbolic mass transport for the second order equation (2.3).

If  $\mathbf{D} \equiv \mathbf{0}$  then the right-hand-side of equation (4.27) is just  $\Lambda$ , and the system is decoupled into two one-way wave equations, corresponding to right and left moving mass waves which are respectively denoted by  $v^+$  and  $v^-$ . We now discuss the interpretation of the  $v^\pm$  from these decoupled equations. For concreteness we just consider  $v^+$ , and it follows that the right going wave must satisfy

$$(4.28) \quad (\partial_x - \kappa^{-1}\mathbb{K}_-^{-1})v^+ = 0.$$

In the special case of the non-diffusive limit  $\tau \rightarrow \infty$ , with  $v$  fixed, the equation (4.28), with use of (4.19), becomes

$$(\partial_x - v\gamma_- \partial_t)v^+ = 0,$$

which is satisfied by solutions of the form  $v^+(t + xv\gamma_-)$ . Now observe that  $\gamma_\pm = \pm 1$ , for  $\gamma = 0$ , and that  $\gamma_- < 0$  for  $\gamma > 0$ . It then follows that this solution is the well known right going wave

having Galilean translational invariance. In this case the right moving wave front travels with speed  $\gamma_+/v$  and the left moving wave front travels at speed  $\gamma_-/v$ <sup>8</sup>. When considering the more general operator found in (4.28), we cannot expect the solution of this equation to exhibit such solution symmetry because the wave will be *attenuated* as it moves to the right. However we still call solutions that satisfy (4.28) the right moving waves.

When the material properties are such that  $\mathbf{D} \neq \mathbf{0}$  we cannot make this physical interpretation for  $v^\pm$ , however we shall still call such solutions left and right moving waves for convenience. It should be apparent the mathematics still makes sense in that  $v^\pm$  satisfy (4.21).

We can now examine under what conditions the dynamics matrix provides an exact splitting. If the velocity field,  $\bar{v}$ , is independent of  $x$ , and the limit  $\kappa^{-1} \rightarrow 0$  while  $\kappa^{-1}\sqrt{\tau} \rightarrow v$  remains fixed is examined, it is found that the splitting is exact and  $\mathbf{D} = \mathbf{0}$ . This means that the two one-way wave equations are decoupled and can be integrated exactly. Another possibility is that  $x$  moves into a region in which the parameters are homogeneous with  $\bar{v} \equiv 0$  and  $\mathbf{D} = \mathbf{0}$  again. Similar interpretations can be made for left-going waves.

## 5. WAVE PROPAGATORS

For simplicity, in the rest of the paper we consider the domain of the problem to be the quarter plane  $\Omega = \{(x, t) \in \mathbb{R}^2 | 0 \leq x \leq \infty, 0 \leq t \leq \infty\}$ . The mass transport processes within the medium in the half space  $0 \leq x \leq \infty$  are described by equations (4.1). Within the semi-infinite region the material parameters  $\kappa, \tau, \bar{v} \in C^1(\mathbb{R})$ , and with little loss of generality we assume the initial condition  $v^\pm(x, 0) = 0, x \in [0, \infty)$ .

Karlsson (Karlsson, 1996) has derived wave propagators for a dispersive electromagnetic problem. We derive the equations for the wave propagators for the advection problem under consideration in this paper. The wave propagators are linear operators that map a mass concentration,  $v^+(x, t)$ , at one spatial position  $x > 0$ , to another position  $x' > 0$ . The propagators are operators defined by

$$(5.1) \quad v^+(x', t + \zeta(x, x')) = \mathbf{P}^+(x', x)v^+(x, t),$$

$$(5.2) \quad v^-(x', t + \zeta(x, x')) = \mathbf{P}^-(x', x)v^-(x, t),$$

where  $\zeta$  is the wave front propagation time of a mass wave moving from the point  $x$  to  $x'$ ; expressions for this function are given in (5.15). There is no restriction on the relative magnitudes of  $x$  and  $x'$  in the definition of the wave propagators. When  $x' > x$  the propagators map the field  $v^+$  forward in the positive  $x$ -direction, along with the advection, and when  $x' < x$  the propagation is backwards, against the advection and in the negative  $x$ -direction.

The propagators satisfy the properties of a group (see (Karlsson, 1996)), and the groups inverse operator is defined through, for example

$$(5.3) \quad \mathbf{P}^+(x', x)\mathbf{P}^+(x, x') = \mathbb{I}$$

<sup>8</sup>See Footnote 6 on page 13.

or

$$(5.4) \quad (P^+(x', x))^{-1} = P^+(x, x').$$

The properties of linearity, causality and time-translational invariance imply the representation for the propagators is of the form

$$(5.5) \quad P^+(x', x)v^+(x, t) = a(x, x')v^+(x, t) + (P^+(x', x; \cdot) * v^+(x, \cdot))(t),$$

$$(5.6) \quad P^-(x', x)v^+(x, t) = (P^-(x', x; \cdot) * v^+(x, \cdot))(t),$$

where  $P^\pm$  is a kernel function, and the  $*$  operator denotes the temporal convolution

$$(P^\pm(x', x; \cdot) * v^+(x, \cdot))(t) = \int_0^t P^\pm(x', x; s)v^+(x, t-s) ds.$$

The factor  $a$  modifies the wave front, and provides attenuation when  $x' > x$ , and amplification when  $x' < x$ , and its functional form is given in equation (A.6).

Causality requires that  $v^\pm(x, t) \equiv 0$  for  $t \leq \zeta(0, x)$ . In equation (5.5) the positive moving field at some point  $x' > 0$  has been written in two parts. The first part is due to the direct forward/backward propagation of the incident field,  $v^+(x, t)$ , with attenuation/amplification and time retardation/advancement, depending on whether  $x' > x$  or  $x' < x$  respectively, and the second part is due to scattering effects in the region — this is provided by  $P^+ * v^+$ . The other propagation operator in (5.2) provides the mapping between the incident right going wave  $v^+(x, t)$ , and a left going wave at  $x' > 0$ .

From (5.5) and (5.4) it is seen that the kernel  $P^+(x, x'; t)$  for the inverse propagator in (5.4), is related to the propagator kernel  $P^+(x', x; t)$  through the equation

$$a(x, x')P^+(x, x'; t) + a(x', x)P^+(x', x; t) + (P^+(x', x; \cdot) * P^+(x, x'; \cdot))(t) = 0.$$

This equation is a Volterra integral equation of the second kind, so that for appropriately smooth functions the existence of the inverse kernel, given the other kernel, is assured.

Now we shall derive the functional equations that the propagator kernels satisfy. The initial step in the derivation is to differentiate the representations of the propagators with respect to either  $x$  or  $x'$ . Differentiation with respect to  $x'$ , the station where the wave is propagating to, leads to a form of the operators that will be required in the sequel. Differentiation with respect to  $x$ , the point where the wave has propagated from, leads to an equation suitable for invariant imbedding — these will not be discussed further here (see (Karlsson, 1996) for information on this case).

Differentiation of equation (5.5), with respect to  $x'$ , yields

$$(5.7) \quad (D_1 + (\partial_{x'} \zeta(x, x'))D_2)v^+(x', t + \zeta(x, x')) = (\partial_{x'} a(x, x'))v^+(x, t) \\ + (\partial_{x'} P^+(x', x; \cdot) * v^+(x, \cdot))(t).$$

Then use of the dynamics, (4.27), to rewrite terms on the left-hand-side of (5.7), and on interchanging the left-hand-side with the right-hand-side, leads to

$$(5.8) \quad (\partial_x a(x, x')) v^+(x, t) + (\partial_x P^+(x', x; \cdot) * v^+(x, \cdot))(t) = (\alpha + \kappa^{-1} \mathbb{K}^{-1} + \mathbb{A}) v^+(x', t + \zeta(x, x')) + (\beta + \mathbb{B}) v^-(x', t + \zeta(x, x')) + (\partial_x \zeta(x, x')) \partial_t v^+(x', t + \zeta(x, x')),$$

where the operators  $\mathbb{A}$  and  $\mathbb{B}$  are defined by the convolution operators in (4.27). Furthermore, the use of (4.7) allows the right-hand-side of (5.8) to be written as

$$(5.9) \quad (\alpha + \gamma_- \kappa^{-1} \sqrt{\tau} \partial_t - \frac{1}{2\kappa\sqrt{\chi}} (\mathbb{I} - \mathbb{L}) + \mathbb{A}) v^+(x', t + \zeta(x, x')) + (\beta + \mathbb{B}) v^-(x', t + \zeta(x, x')) + (\partial_x \zeta(x, x')) \partial_t v^+(x', t + \zeta(x, x')),$$

To proceed requires the following lemma.

**Lemma 5.1.** *If  $a_i \in C$ , and  $v \in C^1$  and*

$$(5.10) \quad a_1(x) v(t) + \int_0^t a_2(x, t-s) v(s) ds + a_3(x) \partial_t v(t) + \int_0^t a_4(x, t-s) \partial_t v(s) ds = 0,$$

for all  $t > 0, x, v$ , then  $a_i = 0$ . That is, terms proportional to  $\partial_t v^+(x, t)$ ,  $v^+(x, t)$ , and terms involving convolutions of  $v^+(x, t)$  and  $\partial_t v^+(x, t)$  are independent.

*Proof.* If we consider  $v$  to be constant, then  $\partial_t v = 0$  and

$$(5.11) \quad a_1(x) + \int_0^t a_2(x, s) ds = 0, \quad \forall t > 0, x.$$

In particular, when  $t = 0$  it follows that  $a_1 = 0$  and therefore

$$(5.12) \quad \int_0^t a_2(x, s) ds = 0, \quad \forall t > 0, x,$$

and the continuity of  $a_2$  then implies that  $a_2 = 0$ . A similar argument with  $v = t$  implies that  $a_3 = a_4 = 0$ .  $\square$

Therefore using (5.5) and (5.6) and lemma 5.1 three equations can be obtained from (5.9), and similar considerations of (5.6) yields two further equations. The first two equations are

$$(5.13) \quad \partial_x P^+ = \frac{1}{2\kappa\sqrt{\chi}} (aL + L * P^+ - P^+) + \alpha P^+ + \beta P^- + aA + A * P^+ + B * P^-,$$

$$(5.14) \quad \partial_x P^- - 2v\sqrt{1 + \gamma^2} \partial_t P^- = \frac{1}{2\kappa\sqrt{\chi}} (P^- - L * P^-) + \delta P^- + \tilde{\gamma} P^+ + aC + D * P^- + C * P^+,$$

where the functional dependence of the dynamics  $\{\alpha, \beta, \tilde{\gamma}, \delta\}$ , on  $x'$ , and  $P^\pm(x', x; t)$ ,  $a(x, x')$  on  $x', x, t$  has been implicitly assumed for notational convenience.

The third equation determines the propagation time between two points  $x$  and  $x'$ , which is

$$\begin{aligned} \zeta(x, x') &= - \int_x^{x'} v(s) \gamma_-(s) ds = \int_x^{x'} \frac{v(s)}{\gamma_+(s)} ds, \\ (5.15) \quad &= \int_x^{x'} v(s) (\sqrt{1 + \gamma^2} - \gamma) ds. \end{aligned}$$

This should be compared to the propagation time found in the case with no advection, namely just the integrated slowness (Wall & Olsson, 1997). As  $0 \geq \gamma_- \geq -1$  when  $\gamma > 0$ , it follows that the phase velocity of the wave front is greater than  $1/v$ , as would be expected due to the additive effect of the advective velocity of the embedding medium. The function  $\zeta(x, x')$  does not have an inverse, and it is convenient to define this travel time function in terms of the function

$$(5.16) \quad \zeta(x) = - \int_0^x v(s) \gamma_-(s) ds = \zeta(0, x),$$

with  $\zeta(x, x') = \zeta(x') - \zeta(x)$ . The function  $\zeta(x)$  can be shown to possess an inverse through the inverse function theorem and the fact that  $v/\gamma_+ > 0$ .

The fourth equation determines the wave front attenuation/amplification factor, which has the representation

$$(5.17) \quad a(x, x') = \exp \left( - \int_x^{x'} ([2\kappa(s) \sqrt{\chi(s)}]^{-1} - \alpha(s)) ds \right),$$

and it is to be noted that when  $x' > x$ ,  $0 < a \leq 1$ , and when  $x' < x$ ,  $a \geq 1$ .

The fifth equation specifies the initial conditions for  $P^-$ , which is

$$(5.18) \quad P^-(x', x; 0) + \frac{\tilde{\gamma}a}{2v\sqrt{1 + \gamma^2}} = 0.$$

**5.1. Forward Green Operators.** The forward Green operators provide the mapping of the left-hand boundary condition, at the boundary of the propagation medium, to an interior point  $x'$  by

$$v^+(x', t + \zeta(0, x')) = \mathbb{G}_f^+(x') v^+(0, t), \quad v^-(x', t + \zeta(0, x')) = \mathbb{G}_f^-(x') v^-(0, t).$$

The forward Green operators are linear convolution operators and are related to the propagator operators by

$$\mathbb{G}_f^+(x') = P^+(x', 0), \quad G^-(x) = P^-(x', 0),$$

with kernels given by

$$G_f^+(x', t) = P^+(x', 0; t), \quad G_f^-(x', t) = P^-(x', 0; t).$$

The kernels  $G_f^\pm(x', t)$  satisfy equations (5.13) and (A.5), but with  $x = 0$ . The boundary and initial conditions appropriate for the forward Green kernels are

$$(5.19) \quad G_f^+(0, t) = 0,$$

$$(5.20) \quad G_f^+(x', 0) = \frac{a(0, x')}{2} \int_0^{x'} \left( 2A(s, 0) + \frac{L(s, 0)}{\kappa\sqrt{\chi}} - \frac{\tilde{\gamma}\beta}{v\sqrt{1+\gamma^2}} \right) ds,$$

where equation (5.20) has been obtained by integrating (5.13) along the  $x$ -axis from  $x = 0$  to  $x'$ , for  $t = 0$ . If the semi-infinite half-space is such that all material parameters are homogeneous for  $x > \ell$  then  $G_f^-$  satisfies the extra boundary value

$$(5.21) \quad G_f^-(\ell, t) = 0.$$

**5.2. Transmission Green operators.** The transmission Green operators provide the mapping of the transmitted field at the station  $x = \ell$  in the propagation medium, to an interior point  $0 \leq x' < \ell$ . These operators were first introduced by He (He, 1993), where they were named *compact* Green operators, because for the non-dispersive problem he analyzed their associated kernel functions had compact support. We prefer to name them *transmission* Green functions to distinguish them from the *forward* Green functions previously described.

These operators  $G_t^\pm$ , are the backward operators  $P^\pm(x', \ell)$ , with  $x' < \ell$ , and hence the transmission Green kernels are the kernels  $P^\pm(x', \ell; t)$  and they satisfy (5.13) and (A.5), but with  $x = \ell$ . The boundary and initial conditions appropriate for the forward Green kernels are

$$(5.22) \quad G_t^+(\ell, t) = 0,$$

$$(5.23) \quad G_t^+(x', 0) = \frac{a(\ell, x')}{2} \int_\ell^{x'} \left( 2A(s, 0) + \frac{L(s, 0)}{\kappa\sqrt{\chi}} - \frac{\tilde{\gamma}\beta}{v\sqrt{1+\gamma^2}} \right) ds,$$

$$G_t^-(0, t) = 0,$$

Therefore the solutions of the first order system of partial differential equations (5.13) and (A.5) are continuous along the characteristic curves associated with the system, but may be discontinuous across these curves. From (5.13), it is seen that the characteristic traces are  $t = \text{constant}$  for  $G_t^+$ , and as  $G_t^+(0, t)$  is continuous for all  $t > 0$ , it follows that  $G_t^+$  is continuous in the region  $\{0 < x < \ell, 0 < t < \infty\}$ . However examination of (5.18) shows that any discontinuity in  $v$ ,  $\tilde{\gamma}$ , or  $\gamma$  will be propagated along the characteristic of (A.5). Because we have assumed that the material parameters are continuously differentiable, it follows that  $G_t^-(x, 0)$  is continuous, except possibly at  $x = \ell$ , with a discontinuity of magnitude

$$(5.24) \quad [G_t^-](\ell, 0) = \frac{\tilde{\gamma}(\ell)a(0, \ell)}{2v(\ell)\sqrt{1+\gamma(\ell)^2}},$$

in the direction of increasing  $t$ . This jump in  $G_t^-$  will propagate along the characteristic curve for  $G_t^-$ .



## 6. SIGNAL RECONSTRUCTION

In order to reconstruct the mass concentration signal at  $x = 0$ , it is necessary to relate the physical variables, the concentration  $c$  and the parabolic mass flux  $\kappa^2 \partial_x c$ , to the split variables  $v^\pm$ . It is seen from (4.4) and (4.14), that the right-going wave is given by

$$(6.1) \quad v^+(\ell, t) = \frac{1}{2} \left( c + \frac{\gamma}{\sqrt{1+\gamma^2}} (\mathbb{M} + \mathbb{I}) c - \kappa \mathbb{K}_\chi \partial_x c \right) (\ell, t)$$

For the experimental apparatus we discussed in §1 the blood flow is into assaying tubes. This means that the advection flux is zero, and the diffusion coefficient is many orders of magnitude smaller than in the flow tube; this implies  $\partial_x c(\ell, t) = 0 = \gamma$ ; hence  $v^+(\ell, t) = c(\ell, t)/2$ .

For other flow problems if we are measuring the concentration at the pipe exit, then because there is purely advective flow out the end of the pipe, it follows that  $\partial_x c(\ell, t) = 0$  (Smith, 1988), and (6.1) reduces to

$$(6.2) \quad v^+(\ell, t) = \frac{1}{2} \left( c + \frac{\gamma}{\sqrt{1+\gamma^2}} (\mathbb{M} + \mathbb{I}) c \right) (\ell, t).$$

In general the derivative  $\partial_x c(\ell, t)$  can be estimated in a stable manner using (3.26) and the method of mollification. Therefore  $v^+$  can be readily identified from the measured concentration  $c(\ell, t)$ .

The propagator equations (5.5) and (5.6) form the basis of the signal reconstruction problem, which for the transmission Green kernels are

$$(6.3) \quad v^-(0, t - \zeta(\ell)) = \mathbb{G}_t^- \circ v^+(\ell, t),$$

$$(6.4) \quad v^+(0, t - \zeta(\ell)) = v^+(\ell, t)/a(0, \ell) + \mathbb{G}_t^+ \circ v^+(\ell, t).$$

From (4.4) it follows that  $c = v^+ + v^-$ , and therefore the reconstruction of the signal at  $x = 0$ ,  $c(0, t - \zeta(\ell))$ , can be computed. This reconstruction is well-posed, as the solution of (5.13) and (A.5) for  $\{G_t^+, G_t^-\}$  is a well-posed problem. Because we can consider  $\tau$  as a regularisation parameter, then provided  $\tau > 0$ , it follows that we have regularised the ill-posed parabolic problem. Effectively we have approximated the ill-posed Volterra integral equation of the first kind, which is obtained from the parabolic problem, by a well-posed Volterra integral equation of the second kind, which is associated with the hyperbolic problem.

## 7. DISCUSSION

A discretisation similar to that in (I) will yield solutions to the propagator equations and the signal reconstruction scheme presented in §6. The use of such schemes will be presented in a later paper. The propagators derived in this paper can also be used to solve the problem of reconstructing spatially varying medium parameters such as  $\kappa(x)$ ,  $\bar{v}(x)$ , and  $\tau(x)$ , in an advective-diffusive medium. Such problems have been considered in (I). An alternative solution approach to the signal reconstruction problem considered in this paper is to use the space marching mollification

scheme presented in (Shorten & Wall, 2001). Spatially varying medium coefficients can also be incorporated into such schemes.

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#### APPENDIX A. DIFFERENT SPLITTING

The splitting in §4 is not unique and different splittings may be more suitable than others. What is essential is to ensure that the split fields are identifiable with the measured quantity. As discussed at the end of §4 we state a splitting is exact when the system dynamics  $\mathbf{D} = \mathbf{0}$  under appropriate conditions. The splitting used earlier in this paper is exact if the advection velocity is zero, or the advection is homogenous, and the diffusion term is set to zero. A splitting that is exact for a homogeneous diffusive and advective medium has to include the term  $\kappa^{-2}\bar{v}$  in the  $\mathbf{C}_{22}$  element of the operator matrix  $\mathbf{C}$  in (4.2); we illustrate this alternative splitting in this appendix. The equations for this splitting are similar in form to those found earlier so we just list the differences here.

Thus the preliminary splitting would use component matrices

$$\mathbf{C} = \begin{bmatrix} 0 & 1 \\ (v^2\partial_t^2 + \kappa^{-2}\partial_t) & v^2\bar{v}\partial_t + \kappa^{-2}\bar{v} \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 0 & 0 \\ \partial_x(\bar{v})(\kappa^{-2} + v^2\partial_t) & -2\partial_x(\ln(\kappa)) \end{bmatrix},$$

in (4.2). In the system case, given by (4.3) the corresponding component matrices are

$$\mathbf{C} = \begin{bmatrix} 0 & -\kappa^{-2}(\tau\partial_t + 1) \\ -\partial_t & \bar{v}\kappa^{-2}\tau\partial_t + \bar{v}\kappa^{-2} \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 0 & 0 \\ -\partial_x(\bar{v}) & 0 \end{bmatrix}.$$

The temporal pseudo-differential operator  $\mathbb{K}_{\pm}^{-1}$ , as given in (4.6) for the previous splitting, now is defined through

$$\mathbb{K}_{\pm}^{-1} = \left( \frac{1}{2}\kappa^{-1}\bar{v}\mathbb{I} + \sqrt{\tau}\gamma\partial_t \pm \sqrt{\partial_t(\tau\tilde{\beta}\partial_t + \beta) + \frac{1}{4}\kappa^{-2}\bar{v}^2} \right),$$

see (3.12) for the definition of  $\beta$ , and  $\tilde{\beta}$ . The operator  $\mathbb{K}_{\pm}^{-1}$  has the non-local representation

$$(A.1) \quad \mathbb{K}_{\pm}^{-1}f = \left( \frac{1}{2}\kappa^{-1} \left( \bar{v} \pm \frac{\beta}{v\sqrt{\tilde{\beta}}} \right) \mathbb{I} + \sqrt{\tau}\gamma_{\pm}\partial_t \pm \frac{1}{2\sqrt{\tau\tilde{\beta}}} \mathbb{L} \right) f(t),$$

here the kernel of (4.8) is now

$$(A.2) \quad L(x, t) = \frac{1}{t} \exp\left(\frac{-\beta t}{2\tau\tilde{\beta}}\right) \mathbb{I}_1\left(\frac{-t}{2\tau\tilde{\beta}}\right).$$

Then the operator  $\mathbb{K}_\pm$  is the pseudo-differential operator defined through

$$\mathbb{K}_\pm = \left( \frac{1}{2} \kappa^{-1} \bar{v} + \sqrt{\tau} \gamma \partial_t \pm \sqrt{\partial_t (\tau \tilde{\beta} \partial_t + \beta) + \frac{1}{4} \kappa^{-2} \bar{v}^2} \right)^{-1},$$

which does not have a simple closed form representation; compare this with (4.10).

The inverse of the operator-valued matrix  $P$  is as (4.14) but with the kernel of equation (4.15),  $M$ , given by

$$M(x, t) = \frac{1}{2\tau\tilde{\beta}} \exp\left(\frac{-\beta t}{2\tau\tilde{\beta}}\right) \left[ I_0\left(\frac{t}{2\tau\tilde{\beta}}\right) - I_1\left(\frac{-t}{2\tau\tilde{\beta}}\right) \right].$$

The operator  $\mathbb{K}_\chi$  is as defined in (4.13) with the kernel  $K_\chi$

$$K_\chi(x, t) = \frac{1}{\sqrt{\tau\tilde{\beta}}} \exp\left(\frac{-\beta t}{2\tau\tilde{\beta}}\right) I_0\left(\frac{t}{2\tau\tilde{\beta}}\right).$$

The algebraic properties of the operators are the similar to those shown in § 4.2 except the last two properties now read as

$$\begin{aligned} \mathbb{K}_+^{-1} + \mathbb{K}_-^{-1} &= 2\gamma\sqrt{\tau}\partial_t + \kappa^{-1}\bar{v}\mathbb{I} \\ \mathbb{K}_+^{-1} - \mathbb{K}_-^{-1} &= 2\sqrt{\partial_t (\tau \tilde{\beta} \partial_t + \beta) + \frac{1}{4} \kappa^{-2} \bar{v}^2}. \end{aligned}$$

In the limiting case of pure diffusion  $\bar{v} \rightarrow 0$ ,  $\gamma \rightarrow 0$ ,  $\beta \rightarrow 1$  and  $\tilde{\beta} \rightarrow 1$ , so it follows that the operators  $\mathbb{K}_\pm$ ,  $\mathbb{K}_\pm^{-1}$  reduce to those of (I). Considering the limiting forms of the operators  $\mathbb{K}_\pm^{-1}$  as  $\tau \rightarrow 0$ ; when  $\tau = 0$ , the model equations are parabolic; it is then possible to show that

$$\lim_{\tau \rightarrow 0} \mathbb{K}_\pm^{-1} = \frac{1}{2} \kappa^{-1} \bar{v} \mathbb{I} \pm \partial_t \mathbb{N},$$

where  $\mathbb{N}$  is the convolution operator with the form of (4.13), but replacing the kernel  $K_\tau$  with the kernel

$$N(x, t) = \frac{1}{\sqrt{\pi t}} \exp\left(-\frac{\kappa^{-2} \bar{v}^2}{4}\right) + \frac{1}{2} \kappa^{-1} \bar{v} \operatorname{erf}\left(\frac{1}{2} \kappa^{-1} \bar{v} \sqrt{t}\right).$$

Turning now to the system dynamics of § 4.3, in terms of the material parameters the first part of the system dynamics is

$$(A.3) \quad P^{-1}(\partial_x P) = \begin{bmatrix} -1 & -1 \\ 1 & 1 \end{bmatrix} j(x) \left( \frac{1}{\tau} \mathbb{K}_\chi + \partial_t \mathbb{K}_\chi \right) + \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} (b(x) \partial_t^2 \mathbb{J} + d(x) \partial_t \mathbb{J} + e(x) \mathbb{J}),$$

with  $\mathbb{J}$  represented by the convolution operator (4.25) but with kernel

$$J(t) = \frac{\kappa^2}{2} \exp\left(\frac{-\beta t}{2\tau\tilde{\beta}}\right) \sinh\left(\frac{t}{2\tau\tilde{\beta}}\right).$$

Coefficient	Expression
$b(x)$	$\partial_x(v^2\tilde{\beta})$
$d(x)$	$\partial_x(\kappa^{-2}\beta)$
$e(x)$	$\frac{1}{2}\kappa^{-2}\bar{v}\partial_x(\kappa^{-2}\bar{v})$
$f(x)$	$-\frac{\gamma}{\sqrt{\beta}}\partial_x(\ln \kappa)$
$g(x)$	$-\partial_x(\ln \kappa)$
$h(x)$	$\frac{1}{2\kappa}\partial_x\bar{v}$
$j(x)$	$\frac{\kappa}{2}\partial_x(v\gamma)$

TABLE 3. Identification of parameters  $b - j$  for hyperbolic mass transport in the second order equation case.

In (A.3)  $\partial_t \mathbb{K}_\chi$  denotes the operator

$$(\partial_t \mathbb{K}_\chi f)(t) = \int_0^t \partial_t \mathbb{K}_\chi(x, t-t') f(t') dt' + K_\chi(0) f(t) = \mathbb{K}'_\chi + K_\chi(0) \mathbb{I},$$

and  $\partial_t \mathbb{J}$  denotes the operator

$$(\partial_t \mathbb{J} f)(t) = \int_0^t \partial_t J(x, t-t') f(t') dt' + J(0) f(t) = (\mathbb{J}' + J(0) \mathbb{I}) f = \mathbb{J}' f.$$

Now since  $J(0) = 0$ , it follows  $\partial_t^2 \mathbb{J}$  denotes the operator

$$(\partial_t^2 \mathbb{J} f)(t) = \int_0^t \partial_t^2 J(x, t-t') f(t') dt' + \partial_t J(0) \mathbb{I} + J(0) \partial_t f(t) = (\mathbb{J}'' + \partial_t J(0) \mathbb{I}) f.$$

The spatial coefficients  $b$ ,  $d$ ,  $e$ , and  $j$  in the system dynamics of equation (A.3) are given in Table 3. See also the new parameter values as listed in the second part of Table 4 for the system dynamics as given in (4.27).

Coefficient	Terms of material parameters
$\overline{\alpha}(x)$	$+j(x)K_\chi(0) - b(x)\partial_t J(0) - (f(x) + \tau h(x)K_\chi(0)) + g(x)$
$\overline{\beta}(x)$	$+j(x)K_\chi(0) + b(x)\partial_t J(0) - (f(x) + \tau h(x)K_\chi(0)) - g(x)$
$\overline{\gamma}(x)$	$-j(x)K_\chi(0) + b(x)\partial_t J(0) + (f(x) + \tau h(x)K_\chi(0)) - g(x)$
$\overline{\delta}(x)$	$-j(x)K_\chi(0) - b(x)\partial_t J(0) + (f(x) + \tau h(x)K_\chi(0)) + g(x)$
Kernel	Terms of material parameters
$A(x, t)$	$+j(x)\left(\frac{1}{\tau}K_\chi + \partial_t K_\chi\right) - (b(x)\partial_t^2 J + d(x)\partial_t J + e(x)J) - (h(x)K_\chi + f(x)M + \tau h(x)\partial_t K_\chi)$
$B(x, t)$	$+j(x)\left(\frac{1}{\tau}K_\chi + \partial_t K_\chi\right) + (b(x)\partial_t^2 J + d(x)\partial_t J + e(x)J) - (h(x)K_\chi + f(x)M + \tau h(x)\partial_t K_\chi)$
$C(x, t)$	$-j(x)\left(\frac{1}{\tau}K_\chi + \partial_t K_\chi\right) + (b(x)\partial_t^2 J + d(x)\partial_t J + e(x)J) + (h(x)K_\chi + f(x)M + \tau h(x)\partial_t K_\chi)$
$D(x, t)$	$-j(x)\left(\frac{1}{\tau}K_\chi + \partial_t K_\chi\right) - (b(x)\partial_t^2 J + d(x)\partial_t J + e(x)J) + (h(x)K_\chi + f(x)M + \tau h(x)\partial_t K_\chi)$

TABLE 4. Identification of coefficients and kernels for hyperbolic mass transport within the system dynamics (4.27).

We now examine the different functional forms for the the wave propagators with the new splitting. Equation (5.8) becomes

$$(\partial_{x'} a(x, x'))v^+(x, t) + (\partial_{x'} P^+(x', x; \cdot) * v^+(x, \cdot))(t) = (\overline{\alpha} + \kappa^{-1}\mathbb{K}^{-1} + \mathbb{A})v^+(x', t + \zeta(x, x')) \\ + (\overline{\beta} + \mathbb{B})v^-(x', t + \zeta(x, x')) + (\partial_{x'} \zeta(x, x'))\partial_t v^+(x', t + \zeta(x, x')).$$

Furthermore, the use of (A.1) allows the right-hand-side of this equation to be written as

$$(A.4) \quad \left[ \overline{\alpha} + \kappa^{-1} \left( \frac{1}{2} \kappa^{-1} \left( \overline{v} - \frac{\beta}{v\sqrt{\beta}} \right) \mathbb{I} + \sqrt{\tau} \gamma_- \partial_t - \frac{1}{2\sqrt{\tau\beta}} \mathbb{L} \right) + \mathbb{A} \right] v^+(x', t + \zeta(x, x')) \\ + (\overline{\beta} + \mathbb{B})v^-(x', t + \zeta(x, x')) + (\partial_{x'} \zeta(x, x'))\partial_t v^+(x', t + \zeta(x, x')).$$

Therefore, using (5.5) and (5.6) and Lemma 5.1 three equations can be obtained from (A.4), and similar considerations of (5.6) yields two further equations. The first two equations are

$$\begin{aligned} \partial_{x'} P^+ = \kappa^{-1} \left( \frac{1}{2} \kappa^{-1} \left( \bar{v} - \frac{\beta}{v \sqrt{\tilde{\beta}}} \right) P^+ - \frac{1}{2 \sqrt{\tau \tilde{\beta}}} (L * P^+ + aL) \right) + \bar{\alpha} P^+ + \bar{\beta} P^- \\ + aA + A * P^+ + B * P^-, \end{aligned} \quad (\text{A.5})$$

$$\begin{aligned} \partial_{x'} P^- - 2v \sqrt{1 + \gamma^2} \partial_t P^- = \kappa^{-1} \left( \frac{\kappa^{-1}}{2} \left( \bar{v} + \frac{\beta}{v \sqrt{\tilde{\beta}}} \right) P^- + \frac{1}{2 \sqrt{\tau \tilde{\beta}}} L * P^- \right) + \bar{\delta} P^- + \bar{\gamma} P^+ \\ + aC + D * P^- + C * P^+. \end{aligned}$$

The third and fifth equations are as in § 5 but the fourth equation which determines the wave front attenuation/amplification factor, has the representation

$$(\text{A.6}) \quad a(x, x') = \exp \left( \int_x^{x'} \left[ \bar{\alpha} + \frac{1}{2 \kappa^2(s)} \left( \bar{v}(s) - \frac{\beta(s)}{v(s) \sqrt{\tilde{\beta}(s)}} \right) \right] ds \right),$$

and in a homogeneous medium it is to be noted that when  $x' > x$ ,  $0 < a \leq 1$ , and when  $x' < x$ ,  $a \geq 1$ .

If the diffusive and advective medium material parameters are homogeneous, then it follows from Table 3 that the splitting is exact and  $\mathbf{D} = \mathbf{0}$ . This means that the two one-way wave equations are decoupled and can be integrated exactly.

In this appendix we have shown that the associated operators for this splitting are very similar, but more complicated, to the splitting operators obtained earlier in this paper which is exact for the case when the media is non diffusive but the advective medium material parameters are homogeneous. We note that the entry length obtained when the parameters are homogeneous in (3.29), appears directly in the propagator equations for this alternate splitting.

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